

The Control of high-dimensional Chaos in Time-Delay Systems

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We present the control of the high-dimensional chaos, with possibly a large number of positive Lyapunov-exponents, of unknown time-delay systems to an arbitrary goal dynamics. We give an existence-and-uniqueness theorem for the control force. In the case of an unknown system, a formula to compute a model-based control force is derived. We give an example by demonstrating the control of the Mackey-Glass system towards a fixed point and a Rössler-dynamics.

We treat the control problem for nonlinear time-delay systems. The aim of the control is to bring the system from a given dynamical state to an arbitrary goal dynamics. We give the conditions under which the control force uniquely exists. In the case of an unknown nonlinear delay system, we introduce a procedure to compute the control force with the help of a previously published identification procedure. The results are also applicable in the case of the very high-dimensional chaotic motion, with possibly a large number of positive Lyapunov exponents, typically observed in nonlinear delay systems. Stability towards noise and towards an imperfect modeling is treated in linear response. Additionally, we discuss the control of unstable periodic orbits with the help of a minimization. A numerical example, the control of the Mackey-Glass system from a very high-dimensional chaotic state towards a stationary state or a non-periodic time-varying state (the low-dimensional chaotic motion observed in the Rössler system), is presented in detail.

I. INTRODUCTION

The possibility of controlling unstable periodic orbits (UPOs) of chaotic systems with the help of suitably chosen external perturbances, which vanish in the case of a successful control, initiated a vast research activity. The OGY-method [1] and the Pyragas-method [2] attracted most attention, so far, and have been successfully applied to various experimental situations. While the OGY-method applies a local modeling, the Pyragas-method does not apply any modeling and is restricted by the value of the imaginary part of the Floquet-exponents of the unstable periodic orbit to be controlled [3]. One of the aims of the control of UPOs has been to enable the switching between different states with the help of small perturbations only. In low-dimensional chaotic states (with only one positive Lyapunov exponent), though, the basic frequencies of most of the UPOs are nearly identical and a lot of UPOs are therefore quite similar. It is the advantage of hyperchaotic chaotic states (with more than one positive Lyapunov exponent) to provide for a large variety of UPOs with different basic frequencies. For this reason, the control of UPOs of hyperchaotic chaos is desirable. The successful control of UPOs of high-dimensional chaos with either a single control force [4], or a multitude of control forces [5] has been reported. More specifically, the control of UPOs of time-delay systems also in hyperchaotic regimes has been achieved with a single control force only [2,6]. Despite that remarkable success in specific cases, it remains an open problem if all hyperchaotic states can be controlled by a single control force, and what actually determines the number of necessary control variables. In other words - under which *necessary* and *sufficient* conditions does a (scalar or multi-variate) force, which controls a hyperchaotic state towards a UPO exist? For the latter type of control we use the term 'control of UPOs' throughout this paper.

On the other hand, we use the term 'control' to describe the more general task to force a system to an arbitrary, predefined goal dynamics, where the control force is not vanishing. The control to a predefined goal dynamics is also the more general, as well as the practically more relevant case, since most technical processes, such as manufacturing, chemical engineering, etc. rely on this type of control. It has been treated in engineering and applied mathematics for a long time. While for the control of linear systems a sufficient understanding has been achieved, the control of

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nonlinear systems, in general, remains an open problem [7]. More specifically, the control of linear time-delay systems were summarized by Marshall [8]. Along these lines, the control of chaotic states in ordinary differential equations has been demonstrated [9]. In the case of an unknown time-evolution equation, though, the method has to rely on a modeling, i.e. with the help of Takens-like techniques, and is therefore restricted to low-dimensional chaotic states in typical situations. The purpose of this paper is to show that the very high-dimensional chaotic states [10] of a possibly unknown time-delay system with N components can be controlled to an arbitrary goal dynamics, with N control forces, for the following reasons: (1) a global model of an unknown, N -component time-delay system can be obtained with the help of N time series only, as shown in [11]- [13]. (2) The global model allows to compensate the instability-inducing effects of the time delay by suitably chosen N control forces as will be shown in this paper. The paper is organized as follows: At first we prove under which conditions a unique control force exists, which allows for a stable control of a time-delay system. This result is then used to derive a control scheme for the control of a time-delay system with *unknown* equations where a global model is achieved with an adequate identification procedure. Finally, we show the result of a computer experiment.

II. THE CONTROL OF UNKNOWN DELAY SYSTEMS

Suppose the dynamics of the system to be controlled is determined by a N -component, non-autonomous, time-delayed differential equation

$$\dot{y} = H(y, y_{\tau_0}, F(t)), \quad y_{\tau_0} = y(t - \tau_0), \quad (1)$$

where $y(t) \in \mathcal{D}_1 \subset \mathcal{R}^N$ is the range of the dynamics, $F(t) \in \mathcal{D}_2 \subset \mathcal{R}^N$ is the range of the external force. Suppose the function $H : \mathcal{R}^{3N} \rightarrow \mathcal{R}^N$ is a continuous and invertible in \mathcal{D}_2 with respect to the third argument. The control problem consists in proving the existence of a force $F(t)$, which makes a predefined goal dynamics $z(t)$ a stable solution of eq. (1). In practice, global stability is desirable, since that excludes coexisting dynamical states. This can be achieved by imposing an additional condition on the dynamics $y(t)$ in the form of a first-order, ordinary, non-autonomous differential equation

$$\dot{y} = g[z(t)](y), \quad y(0) = y_0, \quad (2)$$

with $g[z](y) \in \mathcal{D}_2$ for $y(t) \in \mathcal{D}_1$. The function $g : \mathcal{R}^N \rightarrow \mathcal{R}^N$ has a functional dependence on the goal dynamics $z(t)$. We require $z(t)$ to be a globally stable solution of eq. (2): $\dot{z} = gz$, with $y \in \mathcal{D}_1 \rightarrow z$, for $t \rightarrow \infty$. To ensure the stability of eq. (2) towards a small amount of noise (linear response regime), we also require local stability: $\Re(\lambda_i(z(t))) < 0, \forall t, i = 1, \dots, N$, where $\lambda_i(z(t))$ are the time-dependent eigenvalues of the time-dependent Jacobian of system (2). Equating the two requirements upon the dynamics we get a condition for the control force $F(t)$,

$$g[z](y) = H(y, y_{\tau_0}, F(t)). \quad (3)$$

Since we require invertibility of the function H , the control force $F(t)$ uniquely exists for all values of the delay time τ_0 and can be determined via

$$F(t) = H^{-1}(y, y_{\tau_0}, g[z](y)). \quad (4)$$

The control force cancels the instability-inducing effects of the time-delay. The dynamics is only determined by eq. (2), and inherits its stability properties. Therefore, the goal dynamics $z(t)$ is globally stable, and stable in linear response to a small amount of additional noise. But, the stability range might decrease for an increasing value of τ_0 , leading to an increased sensitivity towards noise for an increasing τ_0 .

From the practical point of view it is essential to discuss the control of delay systems, whose time-delay differential equations are not known, but whose dynamics $y(t)$ is accessible to measurement and to an external driving via the control force $F(t)$. For the ease of presentation, we focus on scalar time-delay systems with a single, discrete delay time τ_0 . We emphasize, though, that the presented ideas are also applicable in the case of multi-component time-delay systems, or systems with multiple delay times or a distribution of delay times. The central task is to find a model equation for the dynamics: $F(t) = H_r^{-1}(y, y_{\tau_r}, \dot{y})$, where the function H_r^{-1} and the delay time τ_r are unknown and have to be determined from the dynamics. It has been shown, recently, that a global model of autonomous scalar time-delay systems can be obtained by measuring a scalar component only, even if the dynamics is very high-dimensional chaotic [11]- [13]. The method relies on the detection of nonlinear correlations of the observable, its time derivative and its time-delayed value. For the problem under consideration, a slight modification of this procedure is appropriate. The basic idea is to disturb the system via a time-dependent external force $F(t) \in \mathcal{D}_F$ and measure

the systems reaction $y(t) \in \mathcal{D}_1$, $\dot{y}(t) \in \mathcal{D}'_1$. Then, one has to find the delay time τ_r , for which nonlinear correlations in the form $F(t) = H_r^{-1}(y, y_\tau, \dot{y})$ exist. Adequate measures to detect nonlinear correlations are, i.e., the filling factor [11,12], the maximal correlation [13], or the forecast error [14]. The function H_r^{-1} can be determined with the help of adequate fitting procedures. The control force is then given by

$$F(t) = H_r^{-1}(y, y_{\tau_0}, g[z](y)), \quad (5)$$

with $(y, y_\tau) \in \mathcal{D}_1$, and $g[z](y) \in \mathcal{D}'_1$ if H_r is invertible. There are an infinite number of functions g , which guarantee the existence of the control force. In practice, g can be chosen to meet the purpose best, a very simple choice will be: $g[z](y) = -1/T(y - (z + T\dot{z}))$, with a single control parameter T . If the function H_r is not invertible in the required range, there exists no control force leading to a stable control. with the chosen g . Possibly, control can be achieved, then, by changing g or the action of the control upon the dynamics (via H_r^{-1}). If the function H_r is locally non-unique invertible in the required range, the control force exists in several regimes and we can define a piecewise continuous control force $F(t)$ similar to eq. (5). In this case special care has to be taken for the values (y, y_{τ_0}) for which the derivative of H_r with respect to the third argument vanishes.

Since, in general, the model H_r^{-1} slightly deviates from the actual dynamics $H^{-1} = H_r^{-1} - \epsilon \bar{H}^{-1}$, with ϵ being small, it is crucial to discuss the stability of the control towards the deviation $\epsilon \bar{H}^{-1}$. For the ease of presentation, we restrict our discussion to the case of an additive action of the control force. Then, the dynamics of the system with control is determined by the non-autonomous time-delay equation $\dot{y} = \epsilon \bar{h}(y, y_\tau) + g[z](y)$. Calculating the deviations $\Delta(t)$, with $y \equiv z + \epsilon \Delta$ in linear response, we end up with the differential equation: $\dot{\Delta} = \frac{\partial g}{\partial y}(z, z)\Delta + \bar{h}(y, y_\tau)$. As expected, the quality of the control is directly related to the quality of the model. Deviations from the goal dynamics are proportional to deviations of the model as long as the linear-response approximations are valid. If not, the deviations of the model might lead to a failure of the control. Since we require the goal dynamics $z(t)$ to be a globally stable solution of (2), and H_r^{-1} is a global model of the delayed system, the goal dynamics will, in typical cases, also be a globally stable solution of (1) with the control force (5). This is a huge advantage compared to a control, which only uses a local modeling, where the control might fail because of coexisting attractors; a situation typically encountered in high-dimensional chaotic systems.

III. THE CONTROL OF THE MACKEY-GLASS SYSTEM

In order to demonstrate the control, we apply the scheme to the Mackey-Glass system, $\dot{y} = f(y_{\tau_0}) - y + F(t)$, $f(y_{\tau_0}) = \frac{3y}{1+y_{\tau_0}^{10}}$, in a computer experiment for several values of the delay time τ_0 and several goal dynamics $z(t)$. The model has been proposed to account for the observed large amplitude oscillations of the number of circulating white blood cells of patients suffering from chronic granulocytic leukemia (first Ref. of [19]). We assume the action of the control force $F(t)$ to be additive and to be known. Therefore, it is sufficient to model the system from a time series of the uncontrolled system and subsequently apply the model to compute the control force $F(t)$ via eq. (5). Because of the demonstrational character of the example we skip most of the details of the identification in order to keep the presentation short. At first, we computed a trajectory of the uncontrolled system ($F = 0$) for $\tau_0 = 80.00$ with a Runge-Kutta algorithm of fourth order (transient time: $100 * \tau_0$; length of the trajectory: $100 * \tau_0$ with 10.000 data points). The range has been $\mathcal{D}_1 = [0.20; 2.02]$. The uncontrolled system exhibits a high-dimensional chaotic state. To estimate the delay time we used a filling factor analysis [11]. The τ -dependent filling factor is shown in Fig. 1(a), where the delay time is indicated by a sharp local minimum. The data allow for a non-erroneous estimate $\tau_r = 80.00 \pm 0.01$. In the next step, the function $h_r(y, y_{\tau_r}) = f_r(y_{\tau_r}) - y$ has to be determined. For the fit of the function f_r we used a rational function with a polynomial of sixth order in the nominator and a polynomial of second order in the denominator. The quadratic error has been $1.59 * 10^{-4}$. In Fig. 1(b) the difference of the two functions $\bar{f} = f - f_r$ is shown.

We choose the requirement upon the dynamics (see eq. (2)), g , in a simple linear form,

$$\dot{y} = g[z](y) = -1/T(y - (z + T\dot{z})), \quad (6)$$

with a single control parameter T . The goal dynamics $z(t)$ is a globally stable solution of (6) for $y \in \mathcal{R}$. Therefore, we arrive at a non-autonomous time-delay equation for the controlled system

$$\dot{y} = \bar{f}(y_{\tau_0}) - \frac{1}{T}(y - (z + T\dot{z})). \quad (7)$$

At first we wish to discuss the control of a fixed point, $z(t) = z_0, \dot{z} = 0$, since in this case the action of the control is most obvious. In Fig. 2 we show the control of a chaotic state ($\tau = 80.0$) to the fixed point $z_0 = 1.5$. Note that for

$\tau = 80.0$ the dynamical state of the uncontrolled system is very high-dimensional with 49 positive Lyapunov-exponents and a Kaplan-Yorke Dimension of 86. The deviations from the fixed point, $\delta y := y - z$, are determined by:

$$\delta \dot{y} = \bar{f}(\delta y_{\tau_0} + z_0) - \frac{1}{T} \delta y. \quad (8)$$

The fixed point δy_0 of (8) is given by $\delta y_0 = T \bar{f}(\delta y_0 + z_0)$. Therefore, the magnitude of the deviations from the fixed point z_0 decrease for $T \rightarrow 0$ as shown in Fig. 3(a). According to eq. (8) the stability of the fixed point can be guaranteed for a small enough T as illustrated in Fig. 3(b). For $T \approx 2.0$, the fixed point y_0 undergoes a Hopf bifurcation to an oscillating state.

In Fig. 4 we present the results of the successful control of the Mackey-Glass system from a low-dimensional chaotic state ($\tau_0 = 10$) to a chaotic solution of the Rössler system, $z(t) = x_1(t)$, where $(\dot{x}_1 = 0.2x_1 - x_2; \dot{x}_2 = x_1 - x_3; \dot{x}_3 = \epsilon + 4x_3(x_2 - 2))$. In the case of a control towards a non-stationary goal dynamics $z(t)$, the quality of the control is affected by the time scale of the goal dynamics. Loosely speaking, the parameters of the control has to be adjusted such that the control is faster, than the a the fastest time scale of the goal dynamics. This will be presented in more detail elsewhere.

IV. OUTLOOK AND CONCLUSIONS

Let us shortly comment on the control of UPOs with a vanishing control force, which attracted most attention so far in nonlinear dynamics. The results of the paper will be also applicable in this case, if we choose the goal dynamics to be: $z = z^{(T)}$, where $z^{(T)}$ is a periodic orbit of the system of period T , which can be reconstructed from time series with well-known techniques [15]. To state our result clearly, *all* UPOs of a scalar time-delay system, even the ones which are embedded in a very high-dimensional chaotic attractor are accessible to the control with a single control variable only. Furthermore, we would like to emphasize that the above arguments apply to *any* solution of a time-delay system, indicating the possibility of controlling to non-stationary and non-periodic (i.e. transients, high-dimensional chaos) trajectories with a vanishing control force. The advantage of previously published methods for the control of UPOs definitely is some sort of adaptive recognition of UPOs requiring only a minimum of prior information about the system. In the framework of our control method, a control of UPOs of time-delay systems can be achieved by requiring a suitably chosen functional upon the control force $F[z^{(T)}](t)$ to be minimal under the variation of the goal dynamics $z^{(T)}$, i. e. ,

$$\dot{y} = H^{-1}(y, y_{\tau_0}, \inf_{z^{(T)}} \int_{t-D}^t d\tau \|F[z^{(T)}](\tau)\|), \quad (9)$$

where the functional is chosen to be an integration over the magnitude of the control force on an interval $[t, t - D]$. Technically, the minimization can be performed using established nonlinear optimization procedures.

To the authors knowledge there is no mathematical statement linking the number of *necessary* control variables to the invariant quantities of the controlled dynamics (such as attractor dimensions or number of positive Lyapunov exponents). It is still an open question how many control variables are necessary to ensure the unique existence of a control force, which gains the control of hyperchaotic chaotic states. In some cases it has been shown numerically that a single control force is sufficient [2,4,6]; in other cases there is evidence that more than one control force is required [5]. It seems to be a *sufficient* condition, as argued by several authors, to apply a number of control variables equal to the number of positive Lyapunov-exponents. Besides that in practical purposes this turns out to be unhandy, we show in this paper that the high-dimensional chaos of a N -component time-delay system can always be controlled with a N -component control force. Indeed, the number of positive Lyapunov-exponents is, in general, not a meaningful quantity in this context. We believe that the number of "localized nonlinearities" as introduced in [16], which for time-delay systems corresponds to the number of components, might be a more appropriate concept to determine the number of control variables.

Since time delays are common phenomena in nature, as well as in industrial processes, we find a wide range of applications for the control of time-delay systems stemming from such different fields as hydrodynamics [17], laser physics [18], physiology [19], and biology [20]. For instance, we envisage the possibility to suppress the large-amplitude oscillations of the high-Reynolds number turbulence observed in the Rayleigh-Benard system [17], with the time-varying temperature gradient as the control force. The same argument applies for the control of a confined jet [17] with the pressure difference as a control force. Equally, the control of the high-dimensional chaotic states of a nonlinear ring resonator [18], described by the Ikeda model, with the laser pump power as the control force, seems to be possible with our control method. Furthermore, in implementing the control in an adaptive manner, a control for non-stationary time-delay systems is envisaged.

Finally, we summarize the conditions for the controllability of time-delay systems: (1) To control a N -component time-delay system, it is sufficient to apply N control forces. (2) The function $H(y, y_{\tau_0}, F)$ has at least to be locally invertible with respect to F in the required range. If the underlying time-evolution of the delay system is not known, one has to rely on identification procedures, which additionally require: (1) The measurement of N time series. (2) appropriate ranges for the identification. The control is stable against a small amount of noise, as well as structurally stable against small deviations of the model.

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FIGURE CAPTIONS

- Fig. 1:** (a) Filling factor under variation of the delay time τ for 1000 equally sized cubes in a three-dimensional space. (b) The deviation of the fit f_r from the function f . The dotted lines indicate the range \mathcal{D}_1 .
- Fig. 2:** Control of the Mackey-Glass system for $\tau_0 = 80$ to the fixed point $z_0 = 1.5$. Starting from a typical initial condition on the attractor (*transient time: 1000*), the control has been switched on at $t = 200$ (control parameter $T = 1$) as indicated by vertical lines: (a) Time series $y(t)$; (b) Control force $F(t)$.
- Fig. 3:** Control of the Mackey-Glass system under variation of T for $\tau = 10$ (*solid circles*), $\tau = 30$ (*open circles*), $\tau = 50$ (*stars*), and $\tau = 80$ (*crosses*): (a) deviation of the fixed point y_0 from the goal dynamics z_0 ; (b) standard-deviation of the controlled dynamics.
- Fig. 4:** (a) Control of the Mackey-Glass system with $\tau_0 = 10$ to a variable of the Rössler system (not shown). The parameter of the control has been $T = 0.10$. (b) Control force $F(t)$. As indicated by lines the control has been switched on at $t = 200$.

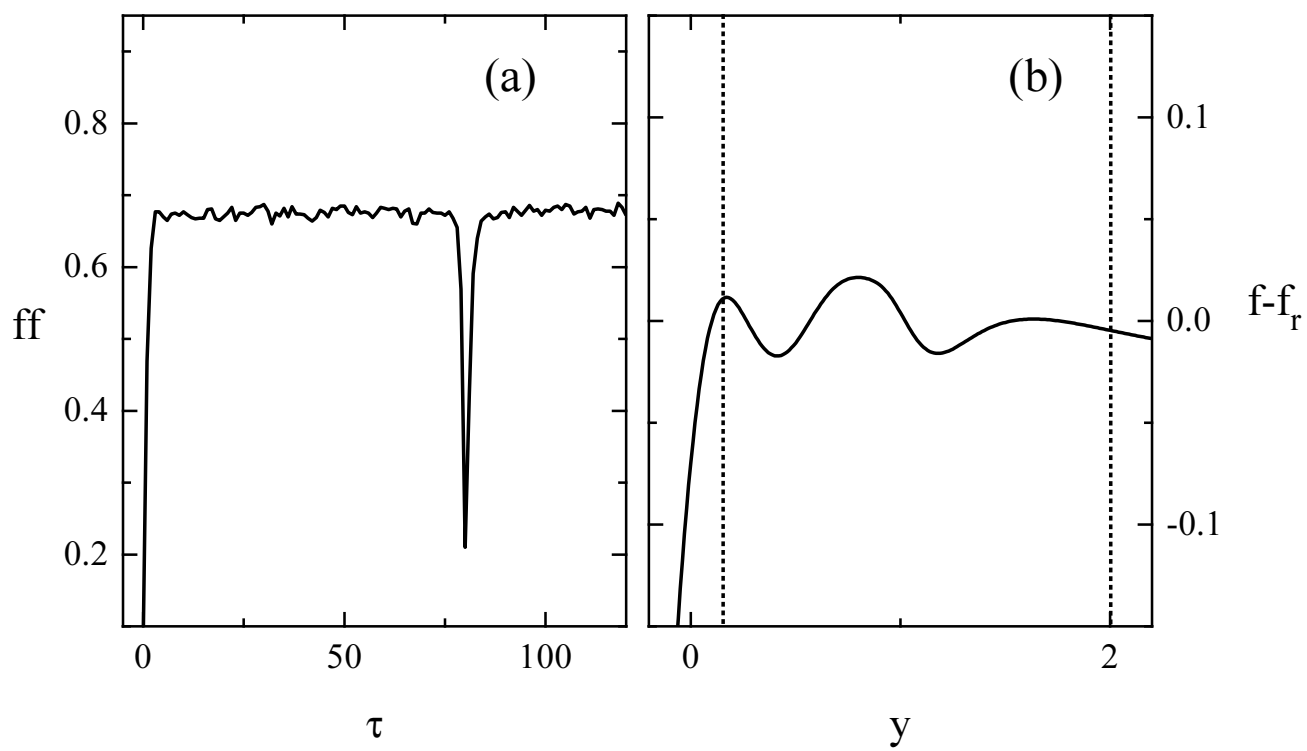


Fig. 1: M. J. Buenner

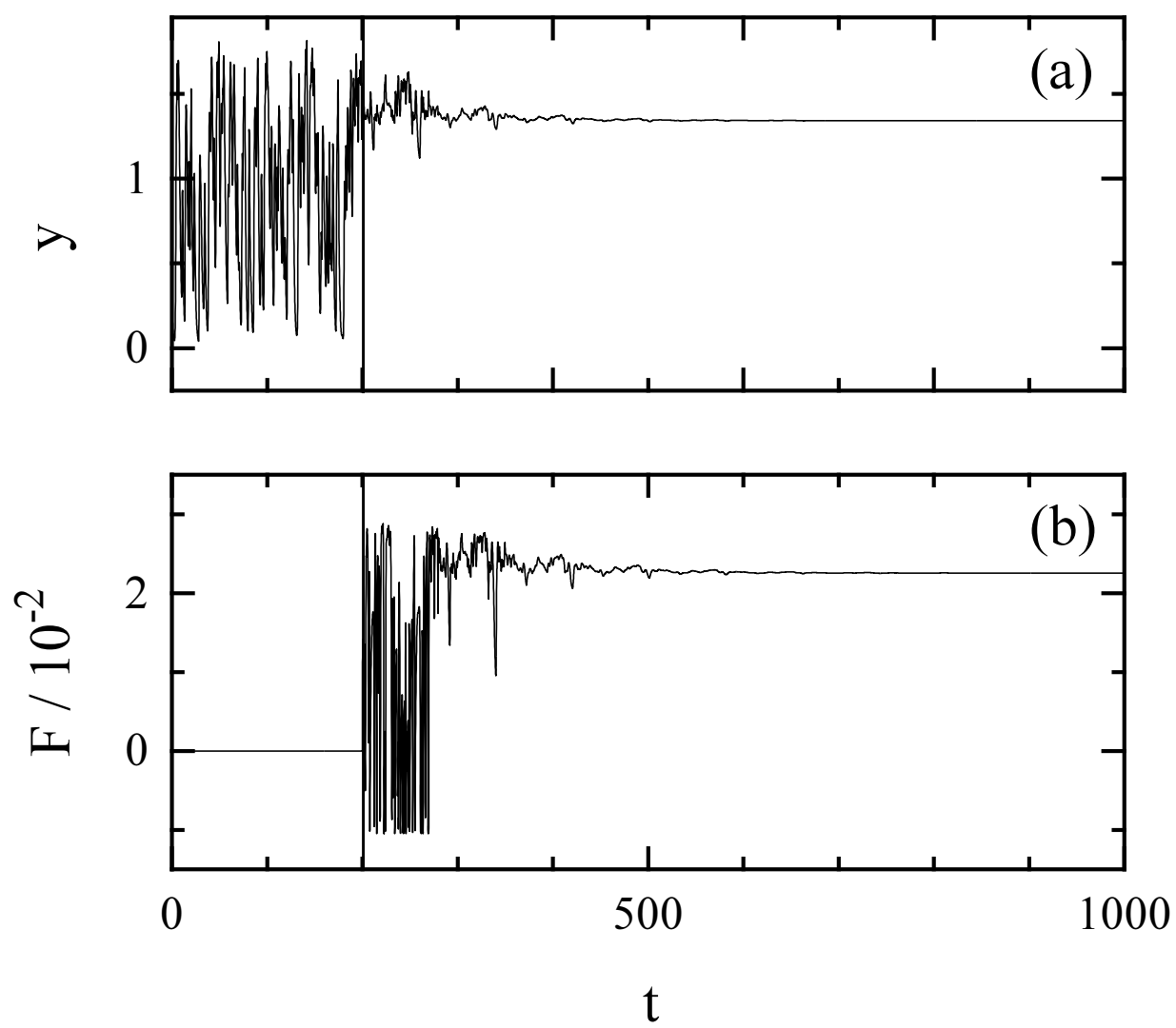


Fig. 2: M. J. Buenner

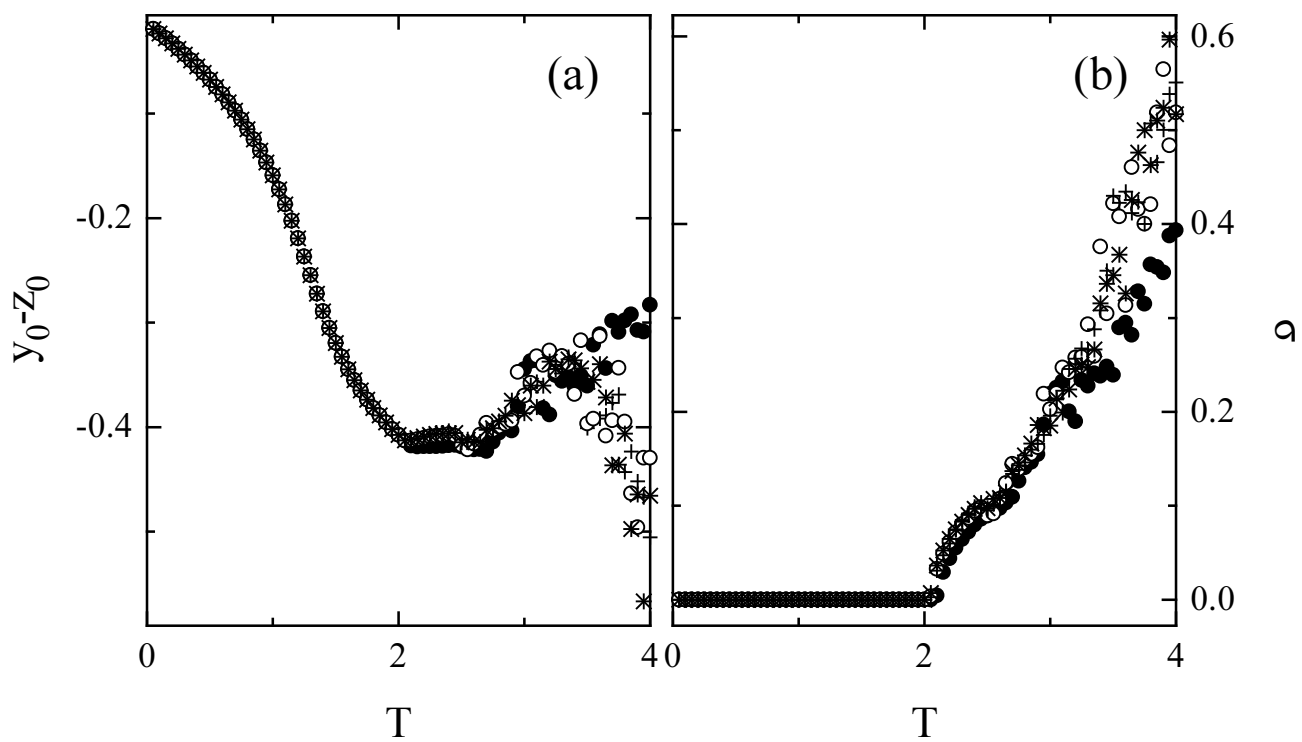


Fig. 3: M. J. Buenner

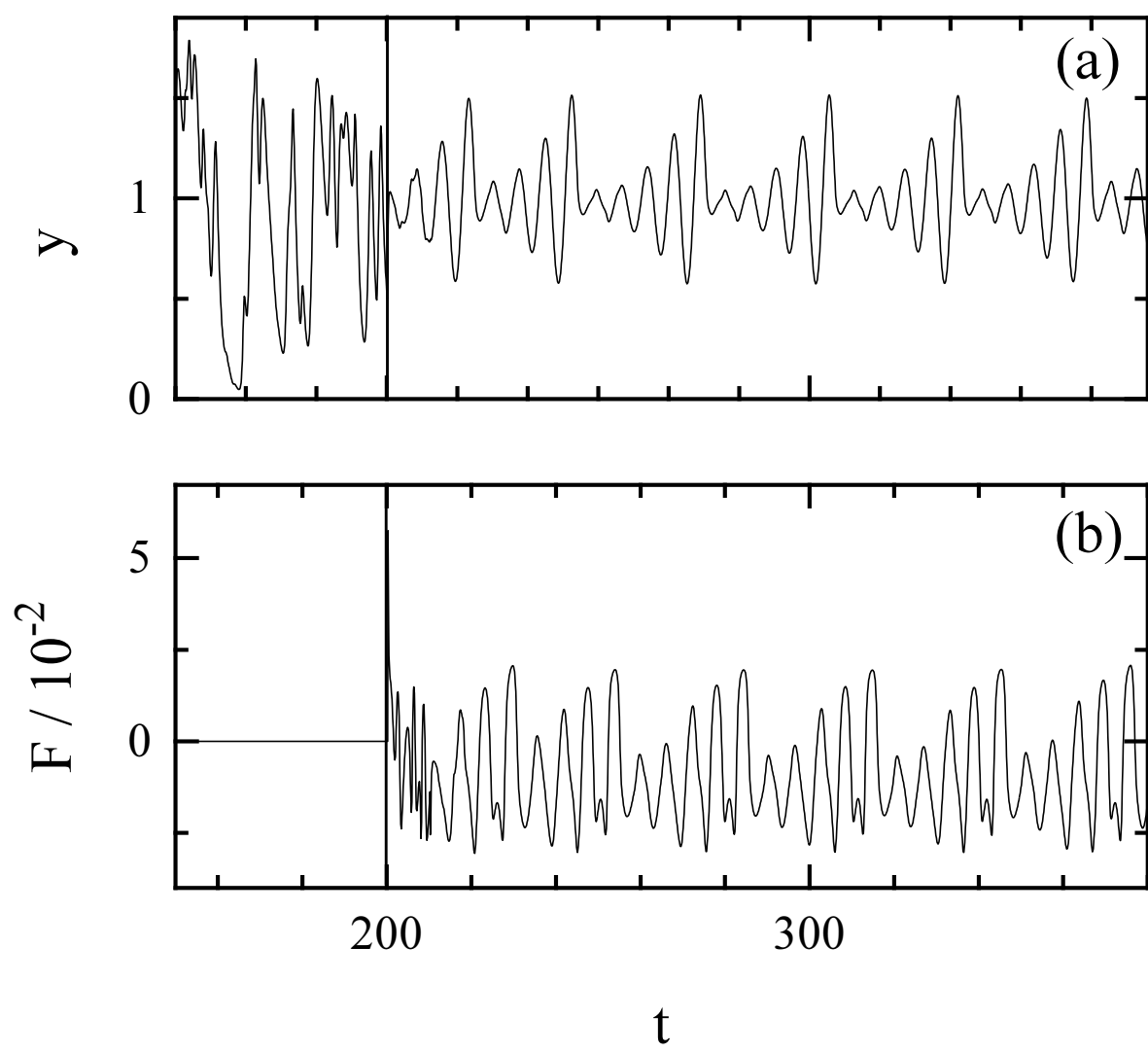


Fig. 4: M. J. Buenner